Interpolacija

Ocjena lokalne pogreške interpolacije za interpolaciju polinomom p:

$$|f(x)-p(x)| \le \frac{|\prod_{i=0}^{n}(x-x_i)|}{(n+1)!} \cdot M_{n+1}(f)$$

Ocjena uniformne pogreške za interpolaciju polinomom p:

$$\max_{x\in[x_0,x_n]} |f(x)-p(x)| \le \begin{cases} \dfrac{\omega(x_0,\ldots,x_n)}{(n+1)!}\cdot M_{n+1}(f), \quad \omega(x_0,\ldots,x_n) = \max\limits_{x\in[x,x_n]} |(x-x_0)\cdot\ldots\cdot(x-x_n)| & \text{općenito} \\ \dfrac{h^{n+1}}{n+1}\cdot M_{n+1}(f) & \text{ekvidistantni čvorovi} \end{cases}$$

Čebiševljevi polinomi:

definicija: $T_0(x)=1, \quad T_1(x)=x, \quad T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$

nultočke od T_n: $x_k=\cos\frac{(2k+1)\pi}{2n}, \quad k=0,\ldots,n-1$

optimalnost: za sve normirane p stupnja n vrijedi $\|p\|_{[-1,1]} \ge \frac{1}{2^{n-1}}\|T_n\|_{[-1,1]}$

Čebiševljevi polinomi na $[a,b]$:

definicija: $\tilde{T}_n(x)=T_n\circ l(x), \quad l(x):[a,b]\to[-1,1]$ rastuća linearna bijekcija

nultočke od $\tilde{T}_n$: $x_k=\frac{1}{2}\left(a+b+(b-a)\cos\frac{(2k+1)\pi}{2n}\right), \quad k=0,\ldots,n-1$

optimalnost: za sve normirane p stupnja n vrijedi $\|p\|_{[a,b]} \ge \|\omega_n\|_{[a,b]} = 2\left(\frac{b-a}{4}\right)^n, \omega_n =$ normirani $\tilde{T}_n(x)$

Ocjena pogreške za interpolaciju linearnim splineom l:

$$\max_{x\in[a,b]}|f(x)-l(x)| \le \frac{1}{8}\Delta^2 M_2(f), \quad \text{za } f\in C^2([a,b])$$

$$\max_{x\in[a,b]}|f(x)-l(x)| \le \frac{1}{2}\Delta M_1(f), \quad \text{za } f\in C^1([a,b])$$

$$|f(x)-l(x)| \le \max_{\xi\in[x_{i-1},x_i]}\{|f(\xi)-f(x_i)|, |f(\xi)-f(x_{i-1})|\}, \quad \text{za } f\in C([a,b]),\ x\in[x_{i-1},x_i]$$

Po dijelovima kubična interpolacija (uz oznake $\varphi|_{[x_{k-1},x_k]}=p_k$, $k=1,\ldots,n$, $p_k(x_{k-1})=f_{k-1}$, $p_k(x_k)=f_k$, $p_k'(x_{k-1})=s_{k-1}$, $p_k'(x_k)=s_k$, $h_k=x_k-x_{k-1}$):

Besselova kvazihermitska: $s_k=\dfrac{h_{k+1}f[x_{k-1},x_k]+h_kf[x_k,x_{k+1}]}{h_k+h_{k+1}}$, $s_0=\dfrac{(2h_1+h_2)f[x_0,x_1]-h_1f[x_1,x_2]}{h_1+h_2}$, $s_n=\dfrac{-h_nf[x_{n-2},x_{n-1}]+(h_{n-1}+2h_n)f[x_{n-1},x_n]}{h_{n-1}+h_n}$

kubična splajn interpolacija: $h_{k+1}s_{k-1}+2(h_k+h_{k+1})s_k+h_ks_{k+1}=3(h_{k+1}f[x_{k-1},x_k]+h_kf[x_k,x_{k+1}]),\ k=1,\ldots,n-1$

Pogreška u Hermiteovoj interpolaciji: postoji $\xi\in\langle x_0,x_m\rangle$ takav da je

$$f(x)-p(x)=\frac{(x-x_0)^{n_0}\cdot(x-x_1)^{n_1}\cdot\ldots\cdot(x-x_m)^{n_m}}{(n+1)!}\cdot f^{(n+1)}(\xi)$$

Integracija

Pogreške u numeričkoj integraciji $R_X = \left|\int_a^b f(x)dx - I_X\right|$:

$$R_T \leq \frac{(b-a)^3}{12} \cdot M_2(f) \quad \text{trapezna} \qquad R_S \leq \frac{(b-a)^5}{2880} \cdot M_4(f) \quad \text{Simpsonova}$$

$$R_{PT} \leq \frac{(b-a)h^2}{12} \cdot M_2(f) \quad \text{produljena trapezna} \qquad R_{PS} \leq \frac{(b-a)h^4}{180} \cdot M_4(f) \quad \text{produljena Simpsonova}$$

Gaussova integracija (A_n je vodeći koeficijent n-tog ortogonalnog polinoma φ_n):

$$\text{težine:} \quad w_k = \frac{A_n}{A_{n-1}} \cdot \frac{\|\varphi_{n-1}\|^2}{\varphi_{n-1}(x_k) \cdot \varphi_n'(x_k)}$$

$$\text{pogreška:} \quad R_n(f) = \frac{\|\varphi_n\|^2}{A_n^2 \cdot (2n)!} \cdot f^{(2n)}(\xi)$$

Pogreška produljene Gauss-Legendreove formule za $\int_a^b f(x)dx \approx \frac{h}{2}\sum_{i=1}^k \sum_{j=1}^n w_j f(g_i(t_j)) + R_{PGL}$ na ekvidistantnoj mreži ($h = \frac{b-a}{k}$):

$$R_{PGL} = \left|\sum_{i=1}^k \frac{h}{2} \cdot R_{n,i}(f)\right| \leq \frac{(n!)^4}{(2n+1)((2n)!)^3} \cdot h^{2n}(b-a)M_{2n}(f)$$

Nelinearne jednadžbe

	Ocjena pogreške:		Kriterij zaustavljanja:
metoda raspolavljanja:	$\lvert\xi - x_n\rvert \leq \frac{1}{2^{n+1}}(b-a), \quad \text{za } f \in C([a,b])$	metoda raspolavljanja:	$n \geq \frac{\log\left(\frac{b-a}{\epsilon}\right)}{\log 2} - 1, \quad \text{za } f \in C([a,b])$
	$\lvert\xi - x_n\rvert \leq \frac{\lvert f(x_n)\rvert}{m_1(f)}, \quad \text{za } f \in C^1([a,b])$		$\lvert f(x_n)\rvert \leq \epsilon \cdot m_1(f), \quad \text{za } f \in C^1([a,b])$
metoda fiksne točke:	$\lvert\xi - x_n\rvert \leq \frac{q^n}{1-q} \cdot \lvert x_1 - x_0\rvert$	metoda fiksne točke:	$n \geq \frac{\log \frac{\epsilon(1-q)}{\lvert x_1 - x_0\rvert}}{\log q}$
	$\lvert\xi - x_n\rvert \leq q^n \cdot \lvert b-a\rvert$		$n \geq \frac{\log \frac{\epsilon}{b-a}}{\log q}$
	$\lvert\xi - x_n\rvert \leq \frac{q}{1-q} \cdot \lvert x_n - x_{n-1}\rvert$		
Newtonova metoda:	$\lvert\xi - x_n\rvert \leq \frac{M_2}{2m_1} \cdot (x_n - x_{n-1})^2$	Newtonova metoda:	$\lvert x_n - x_{n-1}\rvert \leq \sqrt{\frac{2m_1\epsilon}{M_2}}$
metoda sekante:	$\lvert\xi - x_n\rvert \leq \frac{M_1 - m_1}{m_1} \cdot \lvert x_n - x_{n-1}\rvert$	metoda sekante:	$\lvert x_n - x_{n-1}\rvert \leq \frac{\epsilon m_1}{M_1 - m_1}$